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Acoustics of rheologically non-linear solids

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Seismically detectable non-linear effects require corrections to Hooke's model of a solid. Experiments to study the behaviour of various materials point to the dependence that exists between elastic properties and the type of stress. A variable-moduli model takes into consideration the relation of elastic stress to seismic velocities. The paper contains numerical stimulations of velocity profiles and ray paths.

A number of studies using seismic methods have revealed certain phenomena that point to non-linear properties of a solid. The interpretation of seismic results based on Hooke's model fails to account for certain observed effects; for example, time variation of seismic velocities and some others. From lengthy observations at the Nurek hydroelectric power station area (Kulagin and Nikolaev, 1980) and near the Toktogul hydroelectric power station (Silayeva and Terentiev, 1979) the variation of seismic velocities has been shown. The comparison of time variations in the strain of the Earth's surface and seismic velocities around the San-Andreas fault has been shown to be in agreement (Morozova and Nevski, 1984). A number of papers discuss the relation between V_P/V_S variations and processes preceding an earthquake (Slavina et al., 1985). Verbitski (1984) proposed an instrumentation technique for continuous monitoring of the stress state over the Earth's crust based on measuring the variation of seismic velocities.

It is quite possible to take into account the effect of the state of stress of a solid on seismic velocities by using a physically non-linear model of an elastic solid. Experimental studies of the behaviour of various grainy materials show noticeable dependence of elastic properties on the type of loading. It is particularly manifested in the difference between strain characteristics of the material under tension and under compression. Under tension, elastic moduli of the ZTA type of graphite are 20% lower than those under compression (Jones and Nelson, 1976). For cast iron, the Young's modulus under compression is 20% higher than that under tension; for bronze, this value is 10%, and for steel 5% (Ambartsumyan, 1982). The dependence of elastic properties on the type of stress state has been observed in experiments with a variety of design materials (Collins, 1981) and rocks (Zobak and Byerlee, 1975; Alm et al., 1985; Volarovich and Khamidullin, 1985).

Several versions of mechanical variable-moduli models have been suggested; that of Ambartsumyan–Khachatryan (Ambartsumyan, 1982) assumes that in the stress–strain law each of the compliances (ratio of the Poisson coefficient to the Young's modulus) changes its value whenever the stress for which this compliance acts as a factor changes its sign. Jones (Jones and Nelson, 1976) suggested a model of a material with a matrix of weighted compliances; it includes weight coefficients that depend on the absolute values of the principal stresses. For materials with weakly non-linear response, it is possible to assume that the elastic properties depend on the type of stress state. This assumption was the basis for the Rabotnov–Lomakin model (Lomakin and Rabotnov, 1978).

After Lyakhovsky and Myasnikov (1984), the dependence of the elastic potential U on the density ρ, the invariants of the strain tensor ϵ_{ij} and the coefficients λ, μ, ν may be written as

$$U = 1/\rho\left(\lambda/2I_1^2 + \mu I_2 - \nu I_1\sqrt{I_2}\right)$$

$$I_1 = \epsilon_{ii}$$

$$I_2 = \epsilon_{ij}\epsilon_{ij} \qquad (1)$$

The potential (1), apart from second-degree terms I_1^2, I_2 common for elastic solids, includes a non-analytical second-order term $I_1\sqrt{I_2}$. Such an addition to the elastic potential is the simplest for which the requirement of analyticity is rejected. It allows for differing properties of a solid under tension and compression. For models in which non-linearity is taken into account by means of higher order terms, non-linear effects vanish at small strain. The suggested model preserves its properties under arbitrarily small deformations since the principal terms and the additional term are of the same order.

In a solid described by potential (1), the relation of the stress tensor to the strain tensor has the form

$$\sigma_{ij} = \left(\lambda I_1 - \nu\sqrt{I_2}\right)\delta_{ij} + \left(2\mu - \nu I_1/\sqrt{I_2}\right)\epsilon_{ij} \qquad (2)$$

Using the function $\xi = I_2/\sqrt{I_2}$ to describe the form of stress state, effective elastic moduli can be introduced by

$$\lambda^e = \lambda - \nu/\xi; \quad \mu^e = \mu - 1/2\nu\xi$$

To find the value ξ, in terms of σ_{ij}, the following equation should be solved

$$\frac{\sigma_{ii}}{\sqrt{S_2}} = \frac{(3\lambda + 2\mu)\xi - 3\nu - \nu\xi^2}{(2\mu - \nu\xi)\sqrt{1 - 1/3\xi^2}} \qquad (3)$$

where $S_2 = \sigma_{ij}\sigma_{ij} - 1/3\sigma_{ii}^2$

Constructing a numerical method to find the strain tensor in terms of a known stress tensor to solve eq. 3, we use Bairstow's method for calculating the roots of a polynomial with real coefficients. In the case of pure shear stress ($\sigma_{ii} = 0$), eq. 3 has the form

$$\nu\xi^2 - (3\lambda + 2\mu)\xi + 3\nu = 0$$

and is solved as

$$\xi = \frac{3\lambda + 2\mu}{2\nu} - \sqrt{\left(\frac{3\lambda + 2\mu}{2\nu}\right)^2 - 3}$$

The sign in front of the radical is chosen to give regularity of the solution as $\nu \to 0$. For small ν, neglecting higher terms, we get

$$\xi = \frac{3\nu}{3\lambda + 2\mu}$$

The obtained expression for I_1 is strictly positive. This means that a shear stress in the solid causes strains that are accompanied by a cubic strain (dilatation). Using this relation, it is easy to calculate the value of I_1

$$I_1 = \frac{3\nu}{2\mu(3\lambda + 2\mu)}\sqrt{\sigma_{ij}\sigma_{ij}}$$

To find I_1 with $\sigma_{ii} \neq 0$, eq. 3 should be solved numerically.

Dilatancy in rocks was first discovered by Bridgeman in 1949. This effect has been studied in many research works. Figure 1 shows the plot of average pressure $p = (2\sigma_1 + \sigma_3)/3$ of Westerly granite under load versus cubic strain of the sample (Schock, 1977). σ_3 is the maximal tension stress, σ_1, σ_2 are compressive stresses. Figure 2 shows the results of analogous simulations using the above described scheme. The elastic moduli were assumed to be $\lambda = \mu$, $\nu = 0.2\lambda$. A compari-

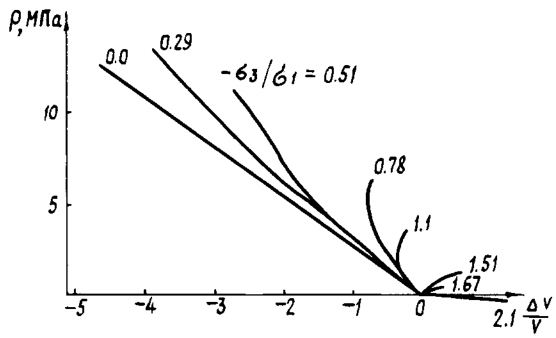

Fig. 1. Average pressure versus cubic strain of Westerly granite sample under load.

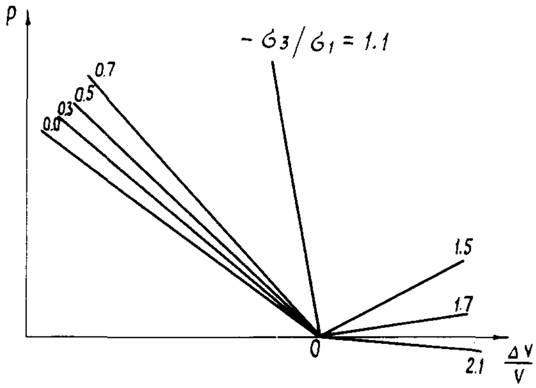

Fig. 2. Simulated average pressure versus cubic strain dependences.

son of the slopes of the average pressure versus cubic strain proves a good agreement between the simulated description of the dilatancy and the experiment.

Notably, there exists a state of stress for which small cubic strains may correspond to a high pressure. Using (2), we write down the expression for σ_{ii}

$$\sigma_{ii} = 3\left(\lambda I_1 - \nu\sqrt{I_2}\right) + \left(2\mu - \nu I_1/\sqrt{I_2}\right) I_1$$

If we set the cubic strain equal to zero ($I_1 = 0$), we get

$$\sigma_{ii} = -3\nu\sqrt{I_2}$$

The corresponding pressure p is given by

$$p = \nu\sqrt{I_2}$$

and is related to purely shear-induced strains and reflects the dilatant properties of the material.

The use of the non-linear relation of stress to strains in the equation of motion

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} \tag{4}$$

adds non-linear terms. In the one-dimensional case, eq. 4 has the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \kappa}\left(\kappa\frac{\partial u}{\partial \kappa} - \nu\left|\frac{\partial u}{\partial \kappa}\right|\right)$$

The general theory of such equations has been considered by Maslov and Mosolov (1985).

In seismic research we can almost always assume that a seismic signal is a small disturbance to the existing stresses. The exception may be a signal propagating close to a vibrator operating at maximum power (Beresnev et al., 1986). For small disturbances of a certain state of strain, described by the tensor ϵ^0_{ij}, we get relations of the form

$$\sigma^1_{ij} = a_{ijkl}\epsilon^1_{kl}$$

where σ^1_{ij}, ϵ^1_{ij} are stress and strain disturbances related to the propagation of a seismic signal, a_{ijkl} is an ϵ^0_{ij}-dependent tensor. The relation between the wave vector k_i and frequency ω is determined by the condition of the determinant equal to zero (Landau and Lifshitz, 1965)

$$\left|a_{ijkl}k_j k_l - \rho\omega^2\delta_{ik}\right| = 0$$

This is a third-degree equation in ω^2 having three distinct roots (Topale, 1985)

$$\begin{aligned}
\omega_P^2 = \frac{\lambda + 2\mu}{\rho}\Bigg[k_i k_i - \frac{\nu}{\lambda + 2\mu} & \times\left\{\frac{I_1}{\sqrt{I_2}}k_i k_i + \frac{2}{\sqrt{I_2}}\epsilon^0_{im}k_i k_m \right. \\
& \left. - \frac{I_1}{(I_2)^{3/2}}\left(\epsilon^0_{im}k_i k_m\right)^2\right\}\Bigg] \\
\omega_{S1}^2 = \frac{\mu}{\rho}\Bigg[k_i k_i\left(1 - \frac{\nu}{\mu}\frac{I_1}{2\sqrt{I_2}}\right) + \frac{\nu}{\mu}\frac{I_1}{(I_2)^{3/2}} & \cdot\left\{\left(k_1\epsilon^0_{2i}k_i - k_2\epsilon^0_{1i}k_i\right)^2 + \left(k_1\epsilon^0_{3i}k_i - k_3\epsilon^0_{1i}k_i\right)^2 \right. \\
& \left. + \left(k_2\epsilon^0_{3i}k_i - k_2\epsilon^0_{2i}k_i\right)^2\right\}\Bigg] \\
\omega_{S2}^2 = \frac{\mu}{\rho}\left(1 - \frac{\nu}{\mu}\frac{I_1}{2\sqrt{I_2}}\right)k_l k_l &
\end{aligned} \tag{5}$$

The existence of three different roots means that there are three types of wave in the solid. With $\nu = 0$, the wave of the first type coincides with a common P-wave, while those of the second and third types coincide with the S-wave. With $\nu \neq 0$ (Topale, 1985), the wave of the first type shows, apart from the longitudinal component, a smaller transverse component (of order ν). The second-type wave adds a small longitudinal component to the transverse one. The third-type wave is purely transverse. Wave velocities depend on the state of strain of the solid. In addition, in the case of the first and the second types of waves, there appears a dependence of velocity on the direction of propagation. The resulting seismic anisotropy is related to the stressed state of the solid. This is discussed at length by Lyakhovsky and Myasnikov (in press).

The equations of the zero-order approximation for the space-time path method in an anisotropic solid have the form (Babich, 1979)

$$v_i = \frac{\partial \omega}{\partial k_i}; \quad \frac{\mathrm{d}x_i}{\mathrm{d}t} = v_i; \quad \frac{\mathrm{d}k_i}{\mathrm{d}t} = -\frac{\partial \omega}{\partial x_i} \tag{6}$$

where x_i are the coordinates of the path points, v_i is the group velocity vector, which is different from $c_i = \omega k_i / k_l k_l$. Choosing one of the expressions (5) as ω, we get equations describing the path of the ray for the given type of wave. For numerical solution of eqs. 6 we used Adam's method with automatic selection of step-length. The value of ν was assumed as 0.05 times the compression modulus.

Some simulated calculations describing the effect of the state of stress on seismic propagation are presented below. The state of stress is set as follows: variant one, hydrostatics ($\sigma_{xx} = \sigma_{zz} = -\rho g h$, $\sigma_{xz} = 0$); variant two: a linearly depth-varying horizontal strain is added to the hydrostatic load. Compressive forces at the surface are 500 bar, and extensional forces at a depth of 30 km are 500 bar. Figures 3 and 4 show velocity profiles and corresponding ray paths. Under non-hydrostatic loads, velocities of waves propagating in the vertical and horizontal directions are different. Owing to the stresses, there is a velocity inversion zone in the second case. This leads to a change in the curvature of the path. For waves propagating in directions close to the horizontal, this zone is a waveguide.

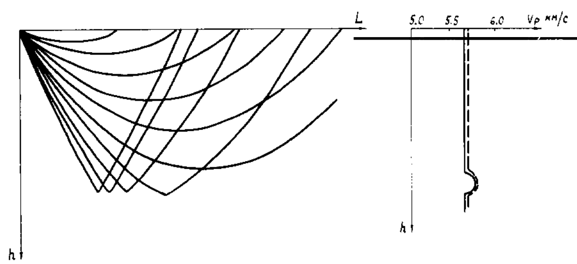

Fig. 3. Velocity distribution and ray paths (variant one): — — — horizontally propagating waves; ——— vertically propagating waves.

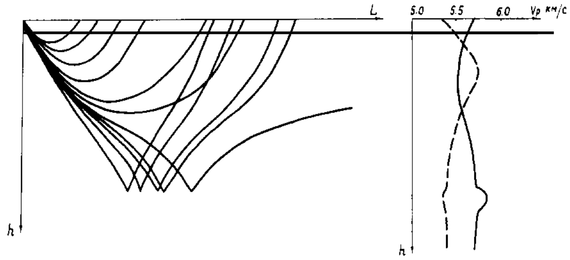

Fig. 4. Velocity distribution and ray paths (variant two). For the legend see Fig. 3.

Figure 5 illustrates wave propagation in a region where ν varies with depth. Both levels, h_1 and h_2 are distinct in the wave field. The upper one acts as a reflector, while the lower one acts as a refractor.

The examples given above show that although the additional coefficient ν is small in relation to the compressive modulus, the resulting dependence of seismic velocities on the stressed state of the solid has a considerable effect on the seismic paths.

To conclude, it should be stressed that the scheme of using a dilatational elastic model of the solid relates the observed data of seismic anisotropy and seismic velocity variation to the physically non-linear properties of the solid. In this case, directly non-linear processes also appear close to a source of seismic waves and around

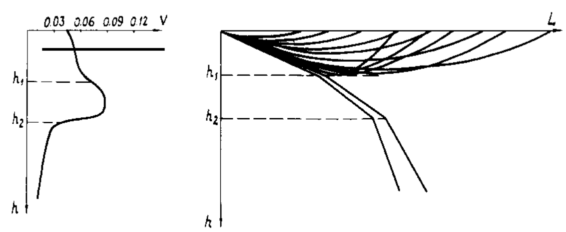

Fig. 5. Distribution of ν and ray paths.

zones free of pre-strain. One of such zones is the free surface. Under existing tectonic stresses outside these zones we can use all the available techniques of linear seismology.

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